Ubiquitously Continuous on Seminorm

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***Abstract*- The term ubiquitously continuous is defined with the purpose of investigating the continuity of a linear operator based on the presence of normed spaces infinite dimensional linear subspaces. Norms and seminorms are parts of the study of normed spaces and obviously are both able to be continuous or discontinuous with respect to their spaces. Hence, this paper aims to explore how ubiquitously continuity of norms and seminorms with respect to infinite dimensional vector spaces. By using available theorems and lemmas, norms and seminorms are generally ubiquitously continuous unless the norms and seminorms themselves are discontinuous at a certain point.**

Keywords— Norm, Seminorm, Ubiquitously Continuous

I. Introduction

Vectors and their operations are fundamental to vector spaces. Any vector space that satisfies the four norm properties can be termed a normed space. On the other hand, when a vector space lacks the positive definite property of a norm, it becomes a seminormed space.

On the other hand, mathematical analysis is closely associated with the term continuity which was discovered by nineteenth-century mathematician Karl Weierstrass. Shortly, the concept of continuity can be explained as how a function behaves normally and there are no sudden unusual changes within that function [1]. The concept of continuity can be developed into the concept of ubiquitously continuous which investigates the implication of how the continuity of a small fraction of a domain causes the domain itself to be continuous or not. Thus, this study seeks to find an answer to how ubiquitously continuity can be applied in any given normed or semi normed space.

A. Vector Spaces

A vector space also a known as a linear space is a nonempty set X defined over a field $\mathbb{F}$ ($\mathbb{R}$ *or* $\mathbb{C}$) that is equipped by two operations, that is vector addition and vector multiplication with scalar [2]. A vector space can also be described as follows:

Definition 1. A vector space over a field $\mathbb{F}$ ($\mathbb{R}$ *or* $\mathbb{C}$) is a nonempty set V that is equipped with two functions that are vector addition from $V \times V$ to V and vector multiplication with scalar $\mathbb{F} \times V$ to V that can be notated by $x + y$ and αx, for all $x, y \in V$, and $\alpha \in \mathbb{F}$, which mean for every $\alpha, \beta \in \mathbb{F}$ and $x, y, z \in V$.

(V1) $x + y = y + x$,
(V2) $x + (y + z) = (x + y) + z$
(V3) There is $0 \in V$ such that $x + 0 = x$
(V4) There is $-x \in V$ such that $x + (-x) = 0$
(V5) $1x = x$
(V6) $\alpha(\beta x) = (\alpha\beta)x$
(V7) $\alpha(x + y) = \alpha x + \alpha y$
(V8) $(\alpha + \beta)x = \alpha x + \beta x$

Example 1. Consider $\mathbb{R}^3$ as a vector space. Let $h, i, j \in \mathbb{R}^n$ and $p, q \in \mathbb{R}^n$ with $h = (h_1, h_2, \dots, h_n), i = (i_1, i_2, \dots, i_n)$ and $j = (j_1, j_2, \dots, j_n)$.

Proof. Now using the properties of the vector spaces, we can get the following:

(V1) $h + i = i + h$,
(V2) $h + (i + j) = (h + i) + j$
(V3) There is $0 \in V$ such that $h + 0 = h$
(V4) There is $-h \in V$ such that $h + (-h) = 0$
(V5) $1h = h$
(V6) $\alpha(\beta h) = (\alpha\beta)h$
(V7) $\alpha(h + i) = \alpha h + \alpha i$
(V8) $(\alpha + \beta)h = \alpha h + \beta h$

Thus, $\mathbb{R}^n$ qualifies as a vector space, because it satisfies all the required properties [3].

B. Normed Spaces

A normed space is a vector space that is provided with a norm. the presence of this norm allows the space to satisfy certain conditions, which define its structure as a normed space [4].

Definition 2. Let X be a vector space over a field $\mathbb{F}$. The norm of X, denoted by $\|\cdot\|: X \to \mathbb{R}$ is a function that associates a real number with each element of X, ensuring the following conditions are satisfied for all $l, m, \in X$, and $\alpha \in \mathbb{F}$:

1. $\|l\| \geq 0$
2. $\|l\| = 0$ if and only if $l = 0$
3. $\|\alpha l\| = |\alpha|\|l\|$

4. $\|l+m\| \leq \|l\|+\|m\|$

Normed space is a vector spaces X that is provided with a norm.

In a normed space, there is also a concept known as the equivalent norm.

Definition 3. [5] A norm $\|\cdot\|$ on a vector space X is considered equivalent to another norm $\|\cdot\|_0$ on X for all $b \in X$ if there exist constants m and n so that

$$m\|b\|_0 \leq \|b\| \leq n\|b\|_0$$

Because of this, the equivalent norms also determine the same topology in the vector space.

Theorem 4. [5] Equivalent norms in X determine the same topology in X.

Example 2. Suppose L is a linear space over field $\mathbb{R}$ by defining $\|l\| = |l_1| + |l_2| + |l_3|$ it will be proved that $\|l\| = |l_1| + |l_2| + |l_3|$, is a norm with $l = (l_1, l_2, l_3)$ where $l \in$ L.

Proof. Based on the conditions of the normed space, we check:

(V1) $\|l\| \geq 0$
(V2) $\|l\| = 0$ if only if $l = 0$
(V3) $\|\alpha l\| = |\alpha|\|l\|$
(V4) $\|l+m\| \leq \|l\|+\|m\|$

Thus, it can be concluded that $(\mathbb{R}, \|\cdot\|)$ is a normed space.

C. Seminorm

A seminorm is a function that maps a vector space to the real numbers, satisfying certain properties [6]. It can be defined as follows:

Definition 5. Let X be a vector space over a field $\mathbb{F}$, a mapping $m: X \to \mathbb{R}$ is named seminorm if

1. $m(x) \geq 0$, for all $x \in X$,
2. $m(x+y) \leq m(x) + m(y)$, for all $x, y \in X$
3. $m(\lambda x) = |\lambda| m(x)$, $f$or all $x \in X$ and all $\lambda \in F$

The function m is always positive, and then m becomes a norm, if $m(x) = 0$ only when $x = 0$.

Example 3. Suppose $x \in \mathbb{R}, m: \mathbb{R} \to \mathbb{R}$ is a function defined by:

$$m(x) = x^2$$

Prove that m is not a seminorm on $\mathbb{R}^2$.

Proof. Based on the conditions of the normed space we get:

(V2) $m(x+y) \leq m(x) + m(y)$, for all $x, y \in X$
The second condition is not stratified because $m(x+y) \leq m(x) + m(y) + 2xy$
(V3) $m(\lambda x) = |\lambda| m(x)$, for all $x \in X$ and all $\lambda \in F$

$$m(\lambda x) = (\lambda x)^2 = \lambda^2 x^2$$

The third condition is not satisfied because of the value of $m(\lambda x) \neq |\lambda| m(x)$.

Since both conditions (V2) and (V3) are not satisfied, so m is not a seminorm.

D. Continuous on Normed Spaces

In normed space, the concept of continuity is explored through equivalent conditions based on Lemma 4 and Definition 5 [7] which state:

Lemma 6. Suppose X and Y are normed spaces, $a \in A \subset X$, and $f: A \to Y$ is a function. Then following statements are equivalent:

1. For every $\varepsilon > 0$, there exists $\delta > 0$ such that if $\|x - a\| < \delta$ then $\|f(x) - f(a)\| < \varepsilon$ where $a \in A$ and $x \in A$.
2. If x_n is a sequence in A and if $x_n \to a$ in X, then $f(x_n) \to f(a)$ in Y.

Definition 7. Suppose X and Y are two normed spaces, $a \in A \subset X$, and a function $f: A \to Y$, then f is considered continuous on A and if f satisfies one of the equivalent conditions based on Lemma 6 If f is continuous on every $a \in A$, then f is said to be continuous on A.

E. Continuous on Seminorms

Definition 8. [9] Let X be vector spaces and m is a seminorm on X. If seminorm m is continuous in one point on X, then seminorm m is continuous everywhere. If X is a normed space accordingly m is continuous on X if only if for all constant number L

$$m(x) \leq L\|x\| \qquad \text{for all } x \in X$$

Then, continuous seminorm on finite dimensional vector spaces can be explained below [10]:

Theorem 9. Let M is a seminorm defined on finite dimensional vector spaces X over the field $\mathbb{F}$, accordingly M is continuous concerning to the distinct topology on X.

F. Ubiquitously Continuous

The term ubiquitously continuous was coined by [11]. Inspired by his work, the term ubiquitously continuous can be defined as below:

Definition 10. Let X and Y is a normed spaces and $L(X, Y)$ represent linear transformation or operator T with $D(T)$ as the linear subspace of the domain, that is X, and $R(T)$ as the range or image, that is Y. For a given linear subspace E define $\mathcal{I}(E)$ as the family of infinite dimensional subspaces of E. An operator T is said to be ubiquitously continuous if for every $E \in \mathcal{I}(X)$ contains an $F \in \mathcal{I}(E)$ for which $T|F$ is continuous.

II. RESULT

A. *Ubiquitously Continuous on Seminorms*

The existence of the origin vector can be used as a tool in order to understand the concept of ubiquitously continuity on norms and seminorms [16].

Lemma 11. Suppose X is an infinite-dimensional vector space over a field $\mathbb{F}$, provided with both a seminorm M and a norm P, accordingly:

a) Continuity of M at $x = 0$ implies ubiquitous continuity concerning to the topology determined by P.
b) Discontinuity of M at $x = 0$ implies ubiquitous continuity concerning to the topology determined by P.

Proof. a) Assume that M is continuous at $x = 0$ concerning to the topology given by the norm P. Therefore, if $(x_n)_{n=1}^{\infty}$ is any sequence in X, so

$$P(x_n) \to 0 \text{ implies } M(x_n) \to 0 \text{ } as \text{ } n \to \infty.$$

Choosing $x \in X$, and taking an arbitrary sequence $(y_n)_{n=1}^{\infty}$ for which

$$x_n = x - y_n \text{ and } P(x - y_n) \to 0,$$

then (1) implies that

$$M(x - y_n) \to 0 \text{ as } n \to \infty.$$

Where for every $x, y \in X$

$$M(x) - M(y) \leq M(x - y) \quad (1)$$

$$M(y) - M(x) \leq M(y - x) \quad (2)$$

So based on (1) and (2) it is obtained that

$$M(y - x) \leq M(x) - M(y) \leq M(x - y).$$

$$|M(x) - M(y)| \leq M(x - y),\ x, y \in X \quad (3)$$

$$|M(x) - M(y_n)| \to 0.$$

Based on the inequality (3)

$$|M(x) - M(y_n)| \leq M(x - y_n) \to 0.$$

Since $M(x - y_n) \to 0$, it can be concluded that

$$|M(x) - M(y_n)| \to 0, n \to \infty.$$

Thus, proving that M is continuous at $x = 0$ means that M is ubiquitously continuous concerning to the topology determined by P.

b) Assuming that M is discontinuous at $x = 0$. There exists a sequence $(x_n)_{n=1}^{\infty}$ in X where $P(x_n) \to 0$ but $M(x_n) \nrightarrow 0$ as $n \to \infty$.

Now that $M(x_n) \nrightarrow 0$ and assume S is a positive, there exists a constant $k > 0$ provided that

$$M(x_n) \geq k,\ n = 1,2,3, \ldots.$$

Since $P(x_n) \to 0$, assume that $P(x_n) \leq \frac{1}{n^2}$, $n = 1,2,3, \ldots$.

$$y_n = nx_n,\ n = 1,2,3, \ldots$$

$$P(y_n) = P(nx_n) = nP(X_n) \leq n\left(\frac{1}{n^2}\right) \leq \frac{1}{n}$$

And

$$M(y_n) = M(nx_n) = nM(x_n) \geq nk$$

Then $P(x_n) \to 0$ but $P(x_n) \to \infty$ as $n \to \infty$.

Therefore, at any point $x \in X$, the sequence $(z_n)_{n=1}^{\infty}$ is defined as

$$z_n = x + y_n,\ n = 1,2,3, \ldots.$$

which satisfies

$$P(x - z_n) = P(y_n) \to 0.$$

However

$$M(x - z_n) = M(y_n) \to \infty$$

which means M is discontinuous at x.

So, it is proved that M is discontinuous at the origin which means it is ubiquitously discontinuous concerning to the topology determined by P.

Theorem 12. Suppose X is an infinite-dimensional vector space over a field $\mathbb{F}$, provided with a seminorm M and a norm P, accordingly:

a) M is ubiquitously continuous in X concerning to the topology determined by P if and only if there exists an arbitrary point in X where M is continuous.
b) Similarly, M is ubiquitously discontinuity in X concerning to the topology determined by P if and only if there exists an arbitrary point in X where M is discontinuous

Proof. a) If there is a point in X for which M is continuous then M is ubiquitously continuous in X concerning to the topology determined by P.

Since there is a point in X for which M is continuous then set the point that is the origin of X. Thus, by the previous Lemma it is clear that M is ubiquitously continuous in X concerning to the topology determined by P.

On the other hand, if M is ubiquitously continuous in X concerning to the topology determined by P then there exists a point in X where M is continuous;

By definition, since M is continuous, then for every E contains F such that $M|F$ is continuous. Consequently, for any chosen E, it is guaranteed that E is continuous. Because E is any subspace of X there exists an E that contains F which is the origin of X. Therefore, by previous proof M is continuous.

Thus, it is proved that M is ubiquitously continuous in X concerning to the topology determined by P if and only if there exists any point in X where M is continuous.

b) If there is a point in X for which M is discontinuous then M is ubiquitously discontinuous;

The definition of ubiquitously continuous states that for every $E \in X$ there is an F contained in E which causes $M|F$ to be continuous. Therefore, it is clear that M must be continuous for every E. However, since there is a point in X where M is discontinuous, such as the existence of the origin vector explained in Lemma, therefore M ubiquitously discontinuous.

If M is ubiquitously discontinuous then there exists a point in X where M is discontinuous;
Similarly, if M is ubiquitously discontinuous then it is implied that $M|F$ is discontinuous. Then, since $F \in E \in X$, clearly that $F \in X$. Therefore, it is true that if M ubiquitously discontinuous then there exists a point in X, precisely F, where M is discontinuous.
Thus, it is proved that M is ubiquitously discontinuous if and only if there exists a point in X for which M is discontinuous.

B. Ubiquitously Continuous on Complete Seminorms

Based on [12], a seminorm M is said to be complete [14] if M has positive (non-zero) values and $M(x) = 0$ for some $x \neq 0$ or namely

$$\ker S = \{x \in X \mid M(x) = 0\}$$

Theorem 13. Suppose P is a norm on an infinite-dimensional vector space X over a field $\mathbb{F}$, accordingly there exists a complete seminorm M that is ubiquitously continuous on X concerning to P.

Proof. Suppose V is a nontrivial finite-dimensional subspace of X and U is also a subspace of X provided that $X = U \oplus V$.
Suppose $P': U \to \mathbb{R}$ is a real valued function measurable concerning to P. The distance from a member in U to a subspace of V is

$$P'(u) = dist_N(u, V) = P(v \in V)P(u - v), \qquad u \in U$$

Since V is infinite-dimensional, the infimum of $P'(u)$ is

$$P'(u) = \min_{v \in V} P(u - v), \qquad u \in U$$

Next, it will be proved that $P'(u)$ is a norm on U with the following conditions:
(N1) $P'(u) = \min_{v \in V} P(u - v) \geq 0$
(N2) $P'(u) = 0$ if and only if $u = 0$.
(N3) $P'(\lambda u) = |\lambda| P'(u)$
(N4) $P'(u_1 + u_2) = P'(u_1) + P'(u_2)$
Thus, it is proved that $P'(u)$ is a norm on U.
Next, for $x \in X$ with decomposition $x = u + v$ where $u \in U$ and $v \in V$, we define

$$M(x) = P'(u)$$

Since M' is a seminorm on U, the mapping $M: X \to \mathbb{R}$ is a seminorm on X. Then, since $\ker M = V$ and V is a complete nontrivial subspace of X, M is a complete seminorm on X.
Furthermore, for any $x = u + v$ in X with $u \in U$ and $v \in V$ we get

$$M(x) = P'(u) = \min_{v' \in V} P(u - v') \leq P(u + v) = P(x)$$

Therefore, for every $x \in X$

$$M(x) \leq P(x)$$

Based on Theorem 7 and Lemma 11 when $M(x) \leq P(x)$ then M is continuous at the origin or $x = 0$ which means M is ubiquitously continuous.

C. Ubiquitously Continuous on Complete and equivalent seminorms

Based on [12], seminorms M_1 and M_2 are equivalent on vector spaces X if there is a constant number $\beta \leq \gamma$ provided that for all $x \in X$.

$$\beta M_1(x) \leq M_2(x) \leq \gamma M_1(x)$$

Corollary 14. [13] For normed spaces X, if $\|\cdot\|$ and $\|\cdot\|'$ is a norm such that both of them a complete normed space and there is $L > 0$ so

$$\|x\| \leq L\|x\|' \text{ for all } x \in X$$

Then, $\|\cdot\|$ and $\|\cdot\|$ is an equivalent norm on normed spaces X.

Proof. Based on identity mapping $\mathcal{I}: (X, \|\cdot\|') \to (X, \|\cdot\|)$.

$$\|x\| \leq L\|x\|' \text{ for all } x \in X$$

Implies that the $\mathcal{I}$ is continuous. If $\|\cdot\|$ and $\|\cdot\|'$ are complete norms or are called Banach spaces then in Banach space [12] a one-to-one linear continuous mapping from Banach space $(X, \|\cdot\|)$ on Banach space $(Y, \|\cdot\|')$ is a topological isomorphism such that there exists $C > 0$ where the mapping $\mathcal{I}^{-1}$ is also continuous which results in

$$\|x\|' \leq C\|x\| \text{ for all } x \in X$$

This shows that $\|\cdot\|$ and $\|\cdot\|'$ are equivalent norms in X because they determine the same topology in X.
Based on that Corollary 14 has been proven by Chmielinski and Goldberg (2020) on [12]:

Theorem 15. [12] Let P be a norm on infinite dimensional vector spaces X with a field $\mathbb{F}$ if P ubiquitously continuous to two complete norms P_1 and P_2 then P_1 and P_2 are equivalent.

So based on that Theorem we can give a counter example about ubiquitously continuous on complete and equivalent seminorms. where perhaps the seminorms are not equivalent as below:

Example 4. We are going to prove that if M is a seminorm on an infinite-dimensional vector space X over $\mathbb{F}$, so M is ubiquitously continuous concerning to two complete seminorms M_1 and M_2. Then M_1 and M_2 maybe not equivalent.
Choose a square n-dimensional linear matrix vector space X that converges to the matrix

$$\bar{X} = \begin{bmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{bmatrix}$$

whose equipped with these seminorms; $M_1(A) = \det(A)$, $M_2(A) = \det(A^T)$, and
$M'(A) \coloneqq \max\{M_1(A), M_2(A)\}, A \epsilon X$.
Then, choose matrix A and B which are

$$A = \begin{bmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \frac{1}{2} \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & \dots & \frac{1}{3} \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{bmatrix}$$

where obviously $A, B \in X$.

Notice that $M_1(A)$, $M_2(A)$, $M_1(B)$, $M_2(B)$, $M_1(A-B)$, and $M_2(A-B)$ are all equal to zero. But, since seminorms are not positive-definite, then, even though

$$M_1(A) = 0 = M_2(A)$$

and

$$M_1(A-B) = 0 = M_2(A-B),$$

it can be concluded that

$$M_1(A) \neq M_2(A).$$

Thus, it has been proven that if M is a seminorm on an infinite-dimensional vector space X over $\mathbb{F}$, so M is ubiquitously continuous concerning to two complete seminorms $M_1$and M_2. Then M_1 and M_2 may not be equivalent.

III. CONCLUSION

The presence of an origin vector can be used in order to check the continuity of norms and seminorms. Consequently, if said norms or seminorms are continuous, then obviously their respective vector spaces can be categorized as vector spaces equipped with ubiquitously continuous norms and seminorms.

Additionally, unlike norms which are definitely equivalent if two complete norms P_1 and P_2 are ubiquitously continuous, the implication may differ if it was done on seminorms. It can be concluded that if M is a seminorm in an infinite dimensional vector space X over field $\mathbb{F}$, if M ubiquitously continuous with concerning two complete seminorms M_1 and M_2, then M_1 and M_2 may or may not be equivalent.

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