Solving Mathematical Models of String Vibrations with Radial Basis Function Networks

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Abstract*-String vibration phenomena on suspension bridges describe oscillations due to external forces and gravity, which can affect the stability and safety of the structure. The mathematical model for this vibration problem is stated by Mckenna (1999) in the form of a second-order ordinary differential equation for deflection $\ddot{y}(t)$. This study aims to analyze the behavior of the $\ddot{y}(t)$ model and solve the model solution with Radial Basis Function (RBF) and compare it with the analytical solution. In RBF, the basis function is selected in the form of a 1st order multiquadratic function. Error analysis is carried out by calculating the difference between RBF and the analytical solution at a time interval $t = [0, 60]$ with parameters $K = 1000, m = 2500, \delta = 0.01, g = 9.8$. The results show that the numerical solution of RBF to the exact solution produces a difference of 0.0628 at time $t = 60$ for $\Delta t = 20$, and a difference of 0.00000007 at time $t = 60$ for $\Delta t = 0.1$. With this level of accuracy, the RBF method can be said to be quite effective in solving string vibration models, and has the potential to be applied to the stability analysis of engineering structures such as suspension bridges.

Keywords—Vibration String Model; Second-Order Differential Equation; Radial Basis Function (RBF); Numerical Solution; Error Analysis

I. INTRODUCTION

Strings play an important role in supporting the bridge load and responding to external forces that affect the string vibration system. Mckenna, (1999) [4] created a mathematical model to describe string vibrations in a suspension bridge system. The purpose of this model is to help understand the string oscillation pattern and its impact on the suspension bridge structure. This model consists of two second-order ordinary differential equations that describe the vertical deflection of the center of gravity (y(t)) as a function of time. The equations are built by considering various physical parameters, such as gravitational acceleration, system mass, spring constant, and viscous damping coefficient against deflection. In addition, vertical and torsional oscillation models, which are formulated using systems of ordinary differential equations, can also be analyzed through numerical methods to observe the dynamic behavior of suspension bridges under different initial conditions [8]. The model used by Mckenna, (1999) is as follows:

$$\ddot{y} = -\delta\dot{y} - \frac{2K}{m}y + g$$

The Radial Basis Function (RBF) method is a numerical technique used to approximate the solution of complex mathematical models. The Radial Basis Function (RBF) method is one of the approaches used in artificial neural networks to solve complex problems, including solving ordinary differential equations (ODEs) [3]. RBF is very suitable for numerical solution of differential equations because it allows the approximation of linear and nonlinear functions. The main advantage of this technique is the ability to directly approximate each function and its derivatives using basis functions without requiring initial values. In this approach, a system of linear equations is built based on the values of the basis functions and their derivatives to calculate the weight values that determine the approximation. In general form, a function that approximates a function $y(t)$ using RBF can be defined as:

$$y(t) = \sum_{j=1}^{N} \omega_j \phi(t, c_j)$$

With $j = 1,2,3, \ldots., N, \omega_j$ is the weight value whose value will be searched and ϕ is the radial basis function (activation function) with various types, such as multiquadratic, Inverse Multiquadratic, or Gaussian [3] [6]. The notation $\phi(t, c_j)$ in this case represents the dependence of the function on the distance between the evaluation point t and the center c_j, namely $\phi(t - c_j)$. In general, the center point c_j is taken the same as the data point t, so that the writing of the RBF function can be simplified to:

$$y(t) = \sum_{j=1}^{N} \omega_j \phi\,(t - c_j)$$

The form of the three radial basis functions is [1]:

1. *Multiquadric* RBF : $\phi(r) = \sqrt{r^2 + \alpha^2}$
2. *Invers Multiquadric* RBF : $\phi(r) = \frac{1}{\sqrt{r^2+\alpha^2}}$
3. *Gaussian* : $\phi(r) = e^{-(\alpha r)^2}$

Where $r = (t - c_j)$ is the distance between the evaluation point t and the center of the radial function c_j. In general, the most commonly used radial basis function is the multiquadratic function, because of its flexibility in handling various forms of data and its ability to provide accurate approximation results.

The RBF function that approaches the continuous solution $y(t)$ can be written as:

$$y(t) = \sum_{j=1}^{N} \omega_j \phi(t, c_j)$$

.

Where $\phi(t, c_j)$ is a radial basis function centered at c_j, and ω_j are the unknown weights (coefficients). For each evaluation point t, the equation produces a system of N linear equations, namely:

$$y(t_1) = \omega_1\phi(t_1, c_1) + \omega_2\phi(t_1, c_2) + \cdots + \omega_N\phi(t_1, c_N)$$
$$y(t_2) = \omega_1\phi(t_2, c_1) + \omega_2\phi(t_2, c_2) + \cdots + \omega_N\phi(t_2, c_N)$$
$$\vdots$$
$$y(t_N) = \omega_1\phi(t_N, c_1) + \omega_2\phi(t_N, c_2) + \cdots + \omega_N\phi(t_N, c_N)$$

Furthermore, the system can be expressed in matrix form as follows [2]. For example:

$$\varphi = \begin{bmatrix} \phi(t_1, c_1) & \phi(t_1, c_2) & \cdots & \phi(t_1, c_N) \\ \phi(t_2, c_1) & \phi(t_2, c_2) & \dots & \phi(t_2, c_N) \\ \vdots & \vdots & \vdots & \vdots \\ \phi(t_N, c_1) & \phi(t_N, c_2) & \cdots & \phi(t_N, c_N) \end{bmatrix}$$

$$W = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \vdots \\ \omega_N \end{bmatrix}$$

$$Y = \begin{bmatrix} y(t_1) \\ y(t_2) \\ \vdots \\ y(t_N) \end{bmatrix}$$

So the system can be written as:

$$\varphi W = Y$$

Or explicitly, the matrix form of the equation is:

$$\begin{bmatrix} \phi(t_1, c_1) & \phi(t_1, c_2) & \cdots & \phi(t_1, c_N) \\ \phi(t_2, c_1) & \phi(t_2, c_2) & \dots & \phi(t_2, c_N) \\ \vdots & \vdots & \vdots & \vdots \\ \phi(t_N, c_1) & \phi(t_N, c_2) & \cdots & \phi(t_N, c_N) \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \vdots \\ \omega_N \end{bmatrix} = \begin{bmatrix} y(t_1) \\ y(t_2) \\ \vdots \\ y(t_N) \end{bmatrix}$$

In numerical methods, errors are an unavoidable part of the process of solving differential equations or other problems solved numerically. Error analysis is important because it provides information about how close the numerical results are to the exact solution.

Errors can be written as:

$$e(t) = |\hat{y}(t) - y(t)|$$

$e(t)$ is the absolute error,
$\hat{y}(t)$ is the exact value of the solution at point t, and
$y(t)$ is the numerical approximation value resulting from the RBF method [5].

In the context of error analysis, the RBF method can provide numerical solutions that approach exact solutions with error levels that are influenced by the type of basis functions used and the parameters in the basis function network.

II. RESEARCH METHODS

A. Types of Research

This research is included in the type of research with a numerical simulation approach. This approach is used to model the string vibration system through numerical solutions of differential equations using the method, namely Radial Basis Function (RBF). This simulation is carried out to determine the effectiveness of the method in solving mathematical equations related to the safety and performance of string vibrations. This research is qualitative, where the expected results are numerical solutions and error analysis of the RBF method.

B. Pre-Research

The pre-research stage begins with a literature study, which examines various literatures relevant to the mathematical model of string vibration, the Radial Basis Function (RBF) method, and the application of the RBF method in the context of ordinary differential equations (ODE), then modeling the string vibration system, which defines the variables and parameters used in the simulation. In this case, the model used is the mathematical model of Mckenna, (1999), where the model will be solved through the numerical solution of the RBF method, which then the solution of the method will be sought for error analysis to determine the effectiveness of the method. After that, software preparation by preparing software to perform numerical simulations such as MATLAB, which supports the computation of the RBF method. Then the determination of initial and boundary conditions, which sets the initial and boundary values for the differential equation of string vibration as input for numerical simulations.

C. Research Data

The mathematical model of string vibrations in this study refers to the journal of Mckenna, (1999), which is stated in the form of a second-order differential equation as follows:

$$\ddot{y} = -\delta\dot{y} - \frac{2K}{m}y + g \tag{1}$$

The initial conditions are given as $y(0) = 12.25$ and $\dot{y}(0) = 1$. In the equation, several parameters have been defined, namely:

K : String spring constant, with a value of 1000 N/m,
m : System mass, with a value of 2500 kg,
δ : Damping coefficient, with a value of 0.01, and
g : Acceleration due to gravity, assumed to be 9.8 m/s^2

.

D. Research Stages

This research stage consists of several steps as follows:

1. String vibration model: The string vibration system is modeled as a differential equation based on predetermined initial and boundary conditions. The mathematical model of string vibration is expressed in the form:

$$\ddot{y} = -\delta\dot{y} - \frac{2K}{m}y + g$$

With initial conditions $y(0) = 12.25$ and $\dot{y}(0) = 1$. This model comes from McKenna, (1999), and contains parameters $K = 1000\ N/m$, $m = 2500\ kg$, $\delta = 0.01$ and $g = 9.8\ m/s^2$.

2. Analytical Solution of Mathematical Model of String Vibration: Solving differential equations using analytical methods with a general approach, namely:
   a. Determining the homogeneous solution $y_h(t)$ by solving the homogeneous differential equation. The approach used is to assume the exponential solution form $y_h(t) = e^{rt}$, then derive and substitute to obtain the characteristic equation.
   b. Solving the characteristic equation to obtain the value of r, which is then used to construct the complete form of the homogeneous solution.
   c. Substituting the initial conditions into the general solution to determine the constants that appear in the homogeneous solution.
   d. Determining the particular solution $y_p(t)$ based on the form of the right-hand side of the equation. Since the right-hand side is a constant, the particular solution is assumed to be a constant C.
   e. Combining the homogeneous and particular solutions to obtain the general solution $y(t) = y_h(t) + y_p(t)$
3. Mathematical Model Solution of String Vibration Using Radial Basis Function (RBF) Method: At this stage, numerical solution is carried out on the above model using the RBF method approach based on multiquadratic functions. Calculations are carried out at two time step values, $\Delta t = 20$ and $\Delta t = 0.1$, to test the effect of time discretization on the results of the numerical solution.
   a. Construction of Approximation Function
   The solution $y(t)$ is assumed to be a linear combination of radial basis functions:

$$y(t) = \sum_{j=1}^{N} \omega_j \phi(t, c_j)$$

   With $\phi(t, c_j) = \sqrt{(t - c_j)^2 + \alpha^2}$, where α is a positive parameter, and c_j is the center point. The first and second derivatives of ϕ are also calculated to obtain the forms $\dot{y}(t)$ and $\ddot{y}(t)$.
   b. Linear Equation System Compilation
   The result of substituting the approximation $y(t)$, $\dot{y}(t)$, and $\ddot{y}(t)$ into the model produces a linear equation system:

$$\sum_{j=1}^{N} \omega_j \left(\phi_{tt}(t, c_j) + \delta \phi_t(t, c_j) + \frac{2K}{m} \phi(t, c_j) \right) = g$$

   For each point t, the system is converted to matrix form:

$$AW = F$$

   With $A = \phi_{tt}(t, c_j) + \delta \phi_t(t, c_j) + \frac{2K}{m} \phi(t, c_j)$, $W = [\omega_1, \omega_2, \dots, \omega_N]^T$, and $F = [g, g, \dots, g]^T$
   c. Linear System Solution
   The solution of the linear system $A \times W = F$ is carried out to obtain the weight vector ω_j, where the matrices A and F are each obtained from the results of multiplying the basis matrix and the target vector. The value of ω_j is calculated using the backslash command (\) in Matlab, namely $W = A \backslash F$, which solves the $A \times W = F$ system through Gaussian elimination.
   d. Calculating the Solution of the String Vibration Model
   After the weight ω_j s obtained, the solution calculation is carried out where the solution $y(t)$ is obtained by multiplying the radial basis function without being derived by the weight values that have been obtained:

$$y(t) = \sum_{j=1}^{N} \omega_j \phi(t, c_j)$$

   For each time point t in the specified interval.
4. Error Analysis: The numerical solution $y(t)$ at both time step values $\Delta t = 20$ and $\Delta t = 0.1$ is compared with the exact solution $\hat{y}(t)$ to evaluate the accuracy of the RBF method. The absolute error is calculated using the formula:

$$e(t) = |\hat{y}(t) - y(t)|$$

   This analysis provides an overview of the effectiveness of the RBF method in solving the string vibration model numerically.
5. Results Visualization: The calculation results at each value of Δt are displayed in the form of graphs depicting the numerical solution $y(t)$, the comparison between the numerical solution and the exact solution, and the absolute error graph $e(t)$. Visualization is done to show the extent to which the numerical approach is able to represent the dynamics of the system accurately.

III. RESULTS AND DISCUSSION

A. Analytical Solution of Mathematical Model of String Vibration

Based on equation 1, the equation of the string vibration model is defined as:

$$\ddot{y}(t) = -\delta \dot{y}(t) - \frac{2K}{m} y(t) + g \qquad (2)$$

With parameters:
$K = 1000\ N/m$
$m = 2500\ kg$
$\delta = 0.01$
$g = 9.8\ m/s^2$
Then Substitute:

$$\frac{2K}{m} = \frac{2(1000)}{2500} = 0.8$$

Then the differential equation becomes:

$$\ddot{y}(t) + 0.01\,\dot{y}(t) + 0.8\,y(t) = 9.8 \qquad (3)$$

With initial conditions:

$$y(0) = 12.25, \dot{y}(0) = 1$$

Then with the substitution and the resulting parameters, it is obtained:

$$y(t) = \frac{200}{31999} e^{-\frac{1}{200}t} \sin\left(\frac{1}{200}\sqrt{31999}t\right)\sqrt{31999} + \frac{49}{4} \qquad (4)$$

For validation and comparison against the results of numerical simulations, analytical solutions are calculated and visualized with two time interval approaches, namely $\Delta t = 20$ and $\Delta t = 0.1$. Evaluation with $\Delta t = 20$ produces 4 evaluation points ($t = 0, 20, 40, 60$), while with $\Delta t = 0.1$ produces a total of 601 evaluation points ($t = 0$ to $t = 60$ with an increase of 0.1). The following is a graph of the results of the analytical solution evaluation at four time points with $\Delta t = 20$ (Fig. 1) and a graph of the results of the analytical solution evaluation at 601 time points with $\Delta t = 0.1$ (Fig. 2):

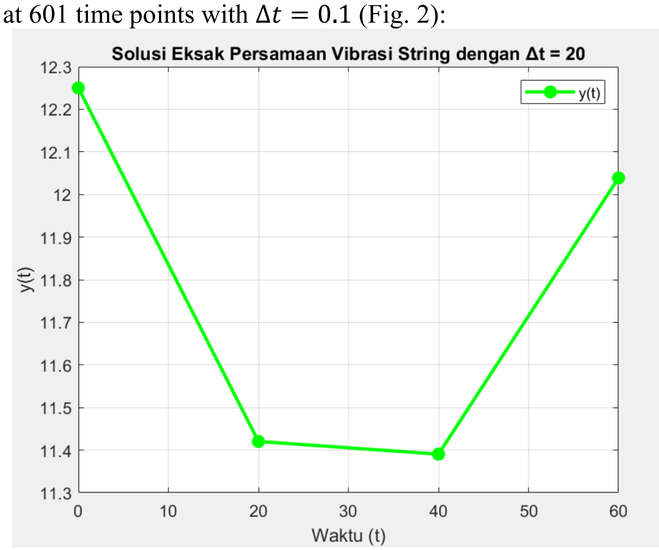

Fig. 1. Exact solution $y(t)$ for $t \in [0,60]$ with initial conditions $y(0) = 12.25$ and $\dot{y}(0) = 1$ with parameters $\Delta t = 20$; $K = 1000$; $m = 2500$; $\delta = 0.01$; $g = 9.8$

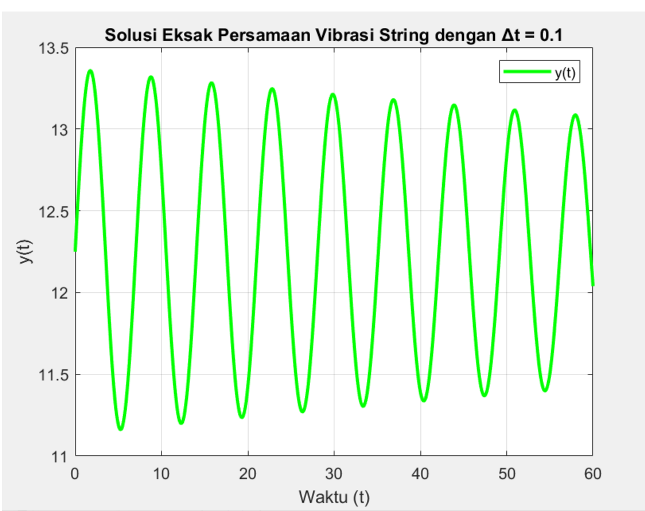

Fig. 2. Exact solution $y(t)$ for $t \in [0,60]$ with initial conditions $y(0) = 12.25$ and $\dot{y}(0) = 1$ with parameters $\Delta t = 0.1$; $K = 1000$; $m = 2500$; $\delta = 0.01$; $g = 9.8$

Based on the graphs in Fig. 1 and Fig. 2, the function $y(t)$ shows an oscillation pattern caused by the influence of system parameters such as K, m, δ, and g. In the graph with $\Delta t = 20$ (Fig. 1), although there are only four evaluation points, the up-down pattern of $y(t)$ can still be observed. The value of $y(t)$ dropped from 12.25 to around 11.4 before rising again to 12. In contrast, the graph with $\Delta t = 0.1$ (Fig. 2) shows a more detailed and smooth oscillation dynamic, showing a more complete waveform in the time range [0, 60].

B. Numerical Solution of Linear Equations of String Vibration Model Using Radial Basis Function Method.

The RBF method basically involves basis functions $\phi(|t - c_j|)$ which depend on the distance between point t and center c_j. For multiquadratic functions, the basis functions are:

$$\phi(r) = \sqrt{r^2 + \alpha^2}$$

In this case, ϕ will be used to construct an approximate solution function in the form of a linear combination of basis functions [6]:

$$y(t) = \sum_{j=1}^{N} \omega_j \phi(t, c_j) \tag{5}$$

Then for $\dot{y}(t)$ and $\ddot{y}(t)$,

$$\phi(t, c_j) = \sqrt{(t - c_j)^2 + \alpha^2} \tag{6}$$

To calculate the first derivative of $\phi_t(t, c_j)$, start with the equation:

$$\phi_t(t, c_j) = \frac{d}{dt}\left(\sqrt{(t - c_j)^2 + \alpha^2}\right)$$

Which can be rewritten as:

$$\phi_t(t, c_j) = \frac{d}{dt}\left((t - c_j)^2 + \alpha^2\right)^{\frac{1}{2}}$$

By using the chain rule which states that the derivative of a composite function is the derivative of the outer function multiplied by the derivative of the inner function [7], it is obtained, resulting in:

$$\phi_t(t, c_j) = \frac{t - c_j}{\sqrt{(t - c_j)^2 + \alpha^2}}$$

So in RBF form, it can be written:

$$\dot{y}(t) = \sum_{j=1}^{N} \omega_j \phi_t(t, c_j) \tag{7}$$

hen for the third derivative, for the derivative of $\phi_t(t, c_j)$ with respect to t can be written

$$\phi_{tt}(t, c_j) = \frac{d}{dt}\left(\frac{t - c_j}{\sqrt{(t - c_j)^2 + \alpha^2}}\right)$$

Then, from the equation, it can be solved using the quotient rule [7] which produces:

$$\phi_{tt}(t, c_j) = \frac{\alpha^2}{\left((t - c_j)^2 + \alpha^2\right)^{\frac{3}{2}}}$$

In RBF form, it can be written:

$$\ddot{y}(t) = \sum_{j=1}^{N} \omega_j \phi_{tt}(t, c_j) \tag{8}$$

Then, from equations (7), (10), and (12), it can be substituted into equation (4) as follows:

$$\sum_{j=1}^{N}\omega_j\phi_{tt}(t,c_j) = -0.01\sum_{j=1}^{N}\omega_j\phi_t(t,c_j) - 0.8\sum_{j=1}^{N}\omega_j\phi(t,c_j) + 9.8$$

And becomes the form of RBF function network:

$$\sum_{j=1}^{N}\omega_j\phi_{tt}(t,c_j) + 0.01\sum_{j=1}^{N}\omega_j\phi_t(t,c_j) + 0.8\sum_{j=1}^{N}\omega_j\phi(t,c_j) = 9.8 \qquad (13)$$

The above equation can be written as:

$$\sum_{j=1}^{N}\omega_j\left(\phi_{tt}(t,c_j) + 0.01\phi_t(t,c_j) + 0.8\phi(t,c_j)\right) = 9.8$$

$$P(t,c_j) = \phi_{tt}(t,c_j) + 0.01\phi_t(t,c_j) + 0.8\phi(t,c_j)$$

$$\sum_{j=1}^{N}\omega_j\left(P(t,c_j)\right) = 9.8$$

C. Numerical Simulation of Linear Equation of String Vibration Model with $\Delta t = 20$ Using Radial Basis Function Method

Based on equation (13), the equation will be partitioned into data with $\Delta t = 20$, where based on the initial value and boundary conditions $t = \{0; 20; 40; 60\}$, is obtained, where the number of known t is 4 ($N = 4$). As for the center points, they are taken from $c = t$. Next, enter the value of t so that it is obtained:

$$9.8 = \omega_1 P(t_1,c_1) + \omega_2 P(t_1,c_2) + \omega_3 P(t_1,c_3) + \omega_4 P(t_1,c_4)$$
$$9.8 = \omega_1 P(t_2,c_1) + \omega_2 P(t_2,c_2) + \omega_3 P(t_2,c_3) + \omega_4 P(t_2,c_4)$$
$$9.8 = \omega_1 P(t_3,c_1) + \omega_2 P(t_3,c_2) + \omega_3 P(t_3,c_3) + \omega_4 P(t_3,c_4)$$
$$9.8 = \omega_1 P(t_4,c_1) + \omega_2 P(t_4,c_2) + \omega_3 P(t_4,c_3) + \omega_4 P(t_4,c_4)$$

With

$$P(t,c_j) = \frac{\alpha^2}{\left((t-c_j)^2+\alpha^2\right)^{\frac{3}{2}}} + 0.01\frac{t-c_j}{\sqrt{(t-c_j)^2+\alpha^2}} + 0.8\sqrt{(t-c_j)^2+\alpha^2}$$

$j = 1, 2, 3, 4$ and α is worth 2.

After that, the system of equations can be changed into a matrix equation as follows:

$$\begin{bmatrix} P(t_1,c_1) & P(t_1,c_2) & P(t_1,c_3) & P(t_1,c_4) \\ P(t_2,c_1) & P(t_2,c_2) & P(t_2,c_3) & P(t_2,c_4) \\ P(t_3,c_1) & P(t_3,c_2) & P(t_3,c_3) & P(t_3,c_4) \\ P(t_4,c_1) & P(t_4,c_2) & P(t_4,c_3) & P(t_4,c_4) \end{bmatrix}\begin{bmatrix}\omega_1\\ \omega_2\\ \omega_3\\ \omega_4\end{bmatrix} = \begin{bmatrix}9.8\\9.8\\9.8\\9.8\end{bmatrix}$$

Then by using the initial conditions $y(0) = 12.25$ and $\dot{y}(0) = 1$, then:

$$\begin{bmatrix} P(t_1,c_1) & P(t_1,c_2) & P(t_1,c_3) & P(t_1,c_4) \\ P(t_2,c_1) & P(t_2,c_2) & P(t_2,c_3) & P(t_2,c_4) \\ P(t_3,c_1) & P(t_3,c_2) & P(t_3,c_3) & P(t_3,c_4) \\ P(t_4,c_1) & P(t_4,c_2) & P(t_4,c_3) & P(t_4,c_4) \\ \phi(0,c_1) & \phi(0,c_2) & \phi(0,c_3) & \phi(0,c_4) \\ \phi_t(0,c_1) & \phi_t(0,c_2) & \phi_t(0,c_3) & \phi_t(0,c_4) \end{bmatrix}\begin{bmatrix}\omega_1\\ \omega_2\\ \omega_3\\ \omega_4\end{bmatrix} = \begin{bmatrix}9.8\\9.8\\9.8\\9.8\\12.25\\1\end{bmatrix}$$

By substituting the values t_1, t_2, t_3, t_4 and c_1, c_2, c_3, c_4 based on $\Delta t = 20$, then:

$$\begin{bmatrix} P(0,0) & P(0,20) & P(0,40) & P(0,60) \\ P(20,0) & P(20,20) & P(20,40) & P(20,60) \\ P(40,0) & P(40,20) & P(40,40) & P(40,60) \\ P(60,0) & P(60,20) & P(60,40) & P(60,60) \\ \phi(0,0) & \phi(0,20) & \phi(0,30) & \phi(0,60) \\ \phi_t(0,0) & \phi_t(0,20) & \phi_t(0,40) & \phi_t(0,60) \end{bmatrix}\begin{bmatrix}\omega_1\\ \omega_2\\ \omega_3\\ \omega_4\end{bmatrix} = \begin{bmatrix}9.8\\9.8\\9.8\\9.8\\12.25\\1\end{bmatrix}$$

Next, calculations are carried out on the matrix $P(t,c_j)$, so that we get:

$$\begin{bmatrix} 2.1 & 16.07034 & 32.03005 & 48.0166 \\ 16.0902 & 2.1 & 16.07034 & 32.03 \\ 32.05 & 16.09024 & 2.1 & 16.0703 \\ 48.0366 & 32.05003 & 16.0902 & 2.1 \\ 2 & 20.0998 & 40.0249 & 60.0167 \\ 0 & -0.995 & -0.999 & -0.9998 \end{bmatrix}\begin{bmatrix}\omega_1\\ \omega_2\\ \omega_3\\ \omega_4\end{bmatrix} = \begin{bmatrix}9.8\\9.8\\9.8\\9.8\\12.25\\1\end{bmatrix}$$

To find the weight value of the matrix equation, a calculation will be carried out through Gaussian elimination, so that:

$$\omega = [0.2094; -0.0157; -0.0127; 0.2096]$$

Then to find the solution of $y(t)$, it can be calculated using the multiplication of the basis function with the weight that has been obtained, so that:

$$\begin{bmatrix}y(t_1)\\ y(t_2)\\ y(t_3)\\ y(t_4)\end{bmatrix} = \begin{bmatrix} \phi(t_1,c_1) & \phi(t_1,c_2) & \phi(t_1,c_3) & \phi(t_1,c_4) \\ \phi(t_2,c_1) & \phi(t_2,c_2) & \phi(t_2,c_3) & \phi(t_2,c_4) \\ \phi(t_3,c_1) & \phi(t_3,c_2) & \phi(t_3,c_3) & \phi(t_3,c_4) \\ \phi(t_4,c_1) & \phi(t_4,c_2) & \phi(t_4,c_3) & \phi(t_4,c_4) \end{bmatrix}\begin{bmatrix}\omega_1\\ \omega_2\\ \omega_3\\ \omega_4\end{bmatrix}$$

$$\begin{bmatrix}y(t_1)\\ y(t_2)\\ y(t_3)\\ y(t_4)\end{bmatrix} = \begin{bmatrix} \phi(0,0) & \phi(0,20) & \phi(0,40) & \phi(0,60) \\ \phi(20,0) & \phi(20,20) & \phi(20,40) & \phi(20,60) \\ \phi(40,0) & \phi(40,20) & \phi(40,40) & \phi(40,60) \\ \phi(60,0) & \phi(60,20) & \phi(60,40) & \phi(60,60) \end{bmatrix}\begin{bmatrix}0.2094\\ -0.0157\\ -0.0127\\ 0.2096\end{bmatrix}$$

$$\begin{bmatrix}y(t_1)\\ y(t_2)\\ y(t_3)\\ y(t_4)\end{bmatrix} = \begin{bmatrix} 2 & 20.099 & 40.050 & 60.033 \\ 20.099 & 2 & 20.099 & 40.050 \\ 40.050 & 20.099 & 2 & 20.099 \\ 60.033 & 40.050 & 20.099 & 2 \end{bmatrix}\begin{bmatrix}0.2094\\ -0.0157\\ -0.0127\\ 0.2096\end{bmatrix}$$

$$\begin{bmatrix}y(t_1)\\ y(t_2)\\ y(t_3)\\ y(t_4)\end{bmatrix} = \begin{bmatrix}12.174\\ 12.314\\ 12.255\\ 12.101\end{bmatrix}$$

Then from this solution, the visualization results of this solution are presented as follows:

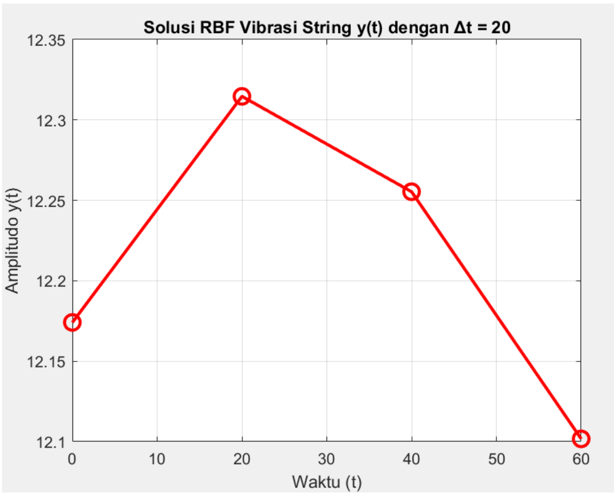

Fig. 3. Graph of Numerical Solution of String Vibration $y(t)$ for $t \in [0,60]$ with initial conditions $y(0) = 12.25$ and $\dot{y}(0) = 1$ with parameters $\Delta t = 20$; $K = 1000$; $m = 2500$; $\delta = 0.01$; $g = 9.8$

Based on the graph in Fig. 3, the numerical solution $y(t)$ is only visualized at four time points, namely $t = 0, 20, 40$, and 60. Although the pattern shown by the points is not as smooth as the graph with a smaller $\Delta t$the graph still shows indications of oscillation. The value is around 12.15–12.2 at $t = 0$, increases to around 12.3 at $t = 20$, drops slightly at $t = 40$, and drops again to 12.1 at $t = 60$. Although the limited number of points makes the oscillation pattern cannot be fully explained, it shows fluctuations in the value of $y(t)$. Therefore, the $\Delta t = 20$ approach is less able to provide a comprehensive picture of the dynamics of the system, but is able to provide a general picture of the oscillation behavior.

D. Numerical Simulation of Linear Equation of String Vibration Model with $\Delta t = 0.1$ Using Radial Basis Function Method

Based on equation (13), the equation will be partitioned into data with $\Delta t = 0.1$, where based on the initial value and boundary conditions $t = \{0; 0.1; 0.2; \ldots; 60\}$, is obtained, where the number of known t is 601 ($N = 601$). As for the center points, they are taken from $c = t$. So by entering the value of t in the equation, we get:

$$9.8 = \omega_1 P(t_1, c_1) + \omega_2 P(t_1, c_2) + \cdots + \omega_{601} P(t_1, c_{601})$$
$$9.8 = \omega_1 P(t_2, c_1) + \omega_2 P(t_2, c_2) + \cdots + \omega_{601} P(t_2, c_{601})$$
$$\vdots$$
$$9.8 = \omega_1 P(t_{601}, c_1) + \omega_2 P(t_{601}, c_2) + \cdots + \omega_{601} P(t_{601}, c_{601})$$

With

$$P(t, c_j) = \frac{\alpha^2}{\left(\left(t - c_j\right)^2 + \alpha^2\right)^{\frac{3}{2}}} + 0.01 \frac{t - c_j}{\sqrt{\left(t - c_j\right)^2 + \alpha^2}} + 0.8\sqrt{\left(t - c_j\right)^2 + \alpha^2}$$

$j = 1, 2, \ldots, 601$ and $\alpha = 2$

So the matrix equation of the equation above is as follows:

$$\begin{bmatrix} P(t_1, c_1) & P(t_1, c_2) & \cdots & P(t_1, c_{601}) \\ P(t_2, c_1) & P(t_2, c_2) & \cdots & P(t_2, c_{601}) \\ \vdots & \vdots & \vdots & \vdots \\ P(t_{601}, c_1) & P(t_{601}, c_2) & \cdots & P(t_{601}, c_{601}) \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \vdots \\ \omega_{601} \end{bmatrix} = 9.8 \tag{9}$$

Then by using the initial conditions $y(0) = 12.25$ and $\dot{y}(0) = 1$, then:

$$\begin{bmatrix} P(t_1, c_1) & P(t_1, c_2) & \cdots & P(t_1, c_{601}) \\ P(t_2, c_1) & P(t_2, c_2) & \cdots & P(t_2, c_{601}) \\ \vdots & \vdots & \vdots & \vdots \\ P(t_{601}, c_1) & P(t_{601}, c_2) & \ddots & P(t_{601}, c_{601}) \\ \phi(0, c_1) & \phi(0, c_2) & \cdots & \phi(0, c_{601}) \\ \phi_t(0, c_1) & \phi_t(0, c_2) & \cdots & \phi_t(0, c_{601}) \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \vdots \\ \omega_{601} \end{bmatrix} = \begin{bmatrix} 9.8 \\ 9.8 \\ \vdots \\ 9.8 \\ 12.25 \\ 1 \end{bmatrix}$$

By substituting the values $t_1, t_2, \ldots, t_N$ and $c_1, c_2, \ldots, c_N$ based on $\Delta t = 0.1$, then:

$$\begin{bmatrix} P(0,0) & P(0,0.1) & \cdots & P(0,60) \\ P(0.1,0) & P(0.1,0.1) & \cdots & P(0.1,60) \\ \vdots & \vdots & \vdots & \vdots \\ P(60,0) & P(60,0.1) & \ddots & P(60,60) \\ \phi(0,0) & \phi(0,0.1) & \cdots & \phi(0,60) \\ \phi_t(0,0) & \phi_t(0,0.1) & \cdots & \phi_t(0,60) \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \vdots \\ \omega_{601} \end{bmatrix} = \begin{bmatrix} 9.8 \\ 9.8 \\ \vdots \\ 9.8 \\ 12.25 \\ 1 \end{bmatrix}$$

After calculating all the matrices in the equation above using Matlab, we get:

$$\begin{bmatrix} 2.1 & 2.0996 & \cdots & 48.0166 \\ 2.1006 & 2.1 & \cdots & 47.9367 \\ \vdots & \vdots & \vdots & \vdots \\ 48.0366 & 47.9567 & \ddots & 2.1 \\ 2 & 2.0024 & \cdots & 60.0334 \\ 0 & -0.0499 & \cdots & -0.9994 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \vdots \\ \omega_{601} \end{bmatrix} = \begin{bmatrix} 9.8 \\ 9.8 \\ \vdots \\ 9.8 \\ 12.25 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} \omega_1 \\ \omega_2 \\ \vdots \\ \omega_{601} \end{bmatrix} = \begin{bmatrix} 2.1 & 2.0996 & \cdots & 48.0166 \\ 2.1006 & 2.1 & \cdots & 47.9367 \\ \vdots & \vdots & \vdots & \vdots \\ 48.0366 & 47.9567 & \ddots & 2.1 \\ 2 & 2.0024 & \cdots & 60.0334 \\ 0 & -0.0499 & \cdots & -0.9994 \end{bmatrix}^{-1} \begin{bmatrix} 9.8 \\ 9.8 \\ \vdots \\ 9.8 \\ 12.25 \\ 1 \end{bmatrix} \tag{10}$$

Based on calculations using backlash (\) in Matlab, the weight values obtained are:

$$\omega = [-2.9113; 2.2088; \cdots; -3.7140]$$

Then to find the solution $y(t)$, it can be obtained by:

$$\begin{bmatrix} y(t_1) \\ y(t_2) \\ \vdots \\ y(t_{601}) \end{bmatrix} = \begin{bmatrix} \phi(t_1, c_1) & \phi(t_1, c_2) & \cdots & \phi(t_1, c_{601}) \\ \phi(t_2, c_1) & \phi(t_2, c_2) & \cdots & \phi(t_2, c_{601}) \\ \vdots & \vdots & \vdots & \vdots \\ \phi(t_{601}, c_1) & \phi(t_{601}, c_2) & \cdots & \phi(t_{601}, c_{601}) \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \vdots \\ \omega_{601} \end{bmatrix} \tag{11}$$

$$\begin{bmatrix} y(t_1) \\ y(t_2) \\ \vdots \\ y(t_{601}) \end{bmatrix} = \begin{bmatrix} \phi(0,0) & \phi(0,0.1) & \cdots & \phi(0,60) \\ \phi(0.1,0) & \phi(0.1,0.1) & \dots & \phi(0.1,60) \\ \vdots & \vdots & \vdots & \vdots \\ \phi(60,0) & \phi(60,0.1) & \dots & \phi(60,60) \end{bmatrix} \begin{bmatrix} -2.9113 \\ 2.2088 \\ \vdots \\ -3.7140 \end{bmatrix}$$

$$\begin{bmatrix} y(t_1) \\ y(t_2) \\ \vdots \\ y(t_{601}) \end{bmatrix} = \begin{bmatrix} 2 & 2.0024 & \cdots & 60.0333 \\ 2.0024 & 2 & \dots & 59.933 \\ \vdots & \vdots & \vdots & \vdots \\ 60.0333 & 59.933 & \cdots & 2 \end{bmatrix} \begin{bmatrix} -2.9113 \\ 2.2088 \\ \vdots \\ -3.7140 \end{bmatrix}$$

$$\begin{bmatrix} y(t_1) \\ y(t_2) \\ \vdots \\ y(t_{601}) \end{bmatrix} = \begin{bmatrix} 12.25 \\ 12.3498 \\ \vdots \\ 12.0388 \end{bmatrix}$$

With graphs:

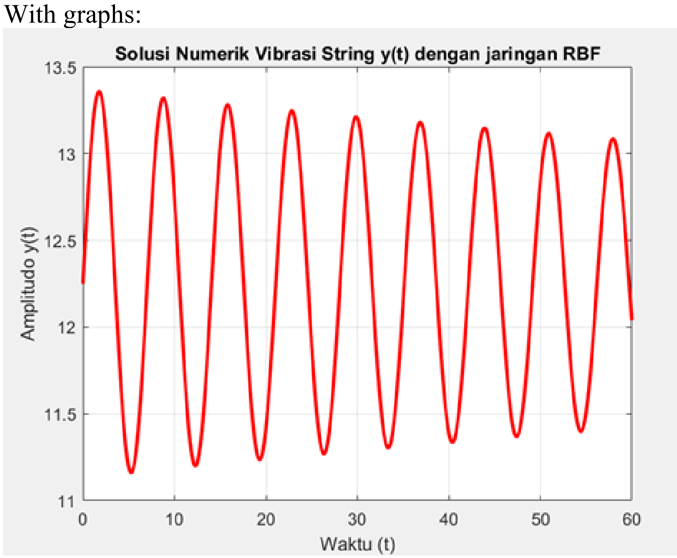

Fig. 4. Graph of Numerical Solution of String Vibration $y(t)$ for $t \in [0,60]$ with initial conditions $y(0) = 12.25$ and $\dot{y}(0) = 1$ with parameters $\Delta t = 0.1$; $K = 1000$; $m = 2500$; $\delta = 0.01$; $g = 9.8$

The solution graph of $y(t)$ shows a stable periodic oscillation pattern with relatively constant amplitude. This indicates that the system compensates for the restoring and damping forces, allowing the oscillations to continue without significant damping. The solution values also indicate that the RBF method is able to capture the dynamic changes of the system and produce a smooth and continuous solution. In this case, the RBF method is also effective in representing the oscillatory behavior of the system, providing an accurate approach to modeling dynamic phenomena, and maintaining the stability of the solution for a longer time.

E. Radial Basis Function Method Error

Error analysis aims to measure the level of accuracy of the numerical solution produced by the Radial Basis Function (RBF) method compared to the exact solution. The error is calculated based on the difference between the numerical solution ($y(t)$) and the exact solution ($\hat{y}(t)$). After that, the results of the numerical solution using the RBF method will be compared with the exact solution. This comparison is visualized in two graphs, each showing the simulation results with $\Delta t = 20$ and $\Delta t = 0.1$ side by side, as shown in Fig. 5 and 6 below:

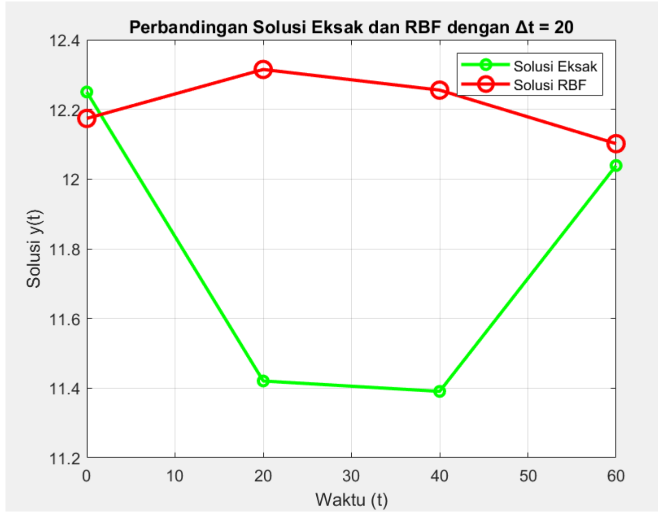

Fig. 5. Comparison of Exact Solution with Numerical Solution of Radial Basis Function with $t \in [0,60]$ and initial condition $y(0) = 12.25$ dan $\dot{y}(0) = 1$ with parameters $\Delta t = 20$; $K = 1000$; $m = 2500$; $\delta = 0.01$; $g = 9.8$

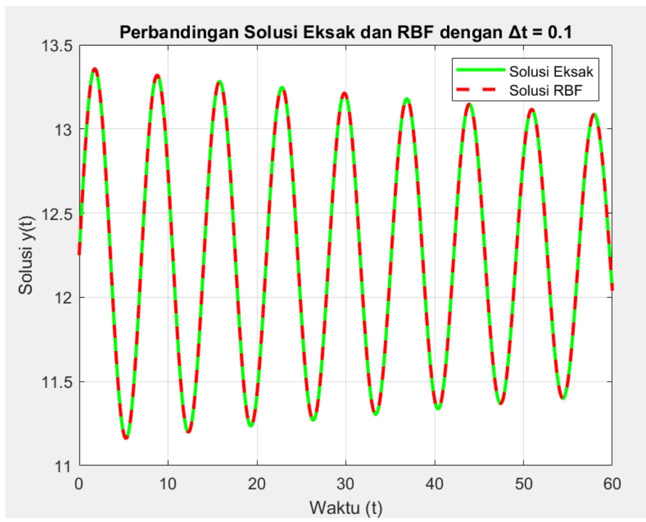

Fig. 6. Comparison of Exact Solution with Numerical Solution of Radial Basis Function with$t \in [0,60]$ and initial condition $y(0) = 12.25$ dan $\dot{y}(0) = 1$ with parameters $\Delta t = 0.1$; $K = 1000$; $m = 2500$; $\delta = 0.01$; $g = 9.8$

The selection of the time interval (t) greatly affects the accuracy of the numerical solution obtained by the RBF method, as shown in the graph in Fig. 5. The exact solution and the RBF solution have significant differences in the left graph $(\Delta t = 20)$. The very small number of evaluation points (N), which is only four points $(t = 0, 20, 40, 60)$ is the main cause of this difference. Due to the limited number of points, the approximation of the solution using the RBF method becomes less accurate and cannot fully capture the dynamics of the system. As a result, the oscillation pattern of the system is unclear and the solution is inaccurate.

After calculating the numerical solution value using the Radial Basis Function (RBF) method, the calculation results are displayed in the form of a table containing time (t), numerical solution ($y(t)$), exact solution ($\hat{y}(t)$), and error analysis results $|\hat{y}(t) - y(t)|$. The calculation results for the time interval $\Delta t = 20$ are shown in Table I. This table presents the solution values at four time points, namely $t = 0, 20, 40,$ and 60, along with the errors between the numerical solution and the exact solution:

TABLE I.
STRING VIBRATION SOLUTION ERROR TABLE $y(t)$ FOR $t \in [0,60]$ WITH INITIAL CONDITIONS $y(0) = 12.25$ AND $\dot{y}(0) = 1$ WITH PARAMETERS $\Delta t = 20$; $K = 1000$; $m = 2500$; $\delta = 0.01$; $g = 9.8$

Time (t)	Exact Solution ($\hat{y}(t)$)	RBF Solution ($y(t)$)	Error $\lvert\hat{y}(t) - y(t)\rvert$
0	12.25	12.174	0.0758
20	11.4205	12.314	0.89413
40	11.3906	12.255	0.86467
60	12.0388	12.101	0.0628

Next, Table II shows the calculation results for the time interval $\Delta t = 0.1$. With a larger number of evaluation points, this table shows the details of the numerical and exact solutions in a tighter time span, so that the accuracy level can be evaluated more thoroughly:

TABLE II.
STRING VIBRATION SOLUTION ERROR $y(t)$ FOR $t \in [0,60]$ WITH INITIAL CONDITIONS $y(0) = 12.25$ AND $\dot{y}(0) = 1$ WITH PARAMETERS $\Delta t = 0.1$; $K = 1000$; $m = 2500$; $\delta = 0.01$; $g = 9.8$

Time (t)	Exact Solution ($\hat{y}(t)$)	RBF Solution ($y(t)$)	Error $\lvert\hat{y}(t) - y(t)\rvert$
0	12.25000000	12.25000000	0.00000000
0.1	12.34981680	12.34981683	0.00000003
0.2	12.44873623	12.44873633	0.00000009
⋮	⋮	⋮	⋮
60	12.03889443	12.03889436	0.00000007

Based on Table I and Table II, it can be seen that the absolute error value $|\hat{y}(t) - y(t)|$ varies at each time point. This shows how the numerical solution approach with the RBF method behaves towards the exact solution, both at the time interval $\Delta t = 20$ and $\Delta t = 0.1$. To provide a visual illustration of the error distribution at both time intervals, the absolute error change pattern is shown side by side in the following graph:

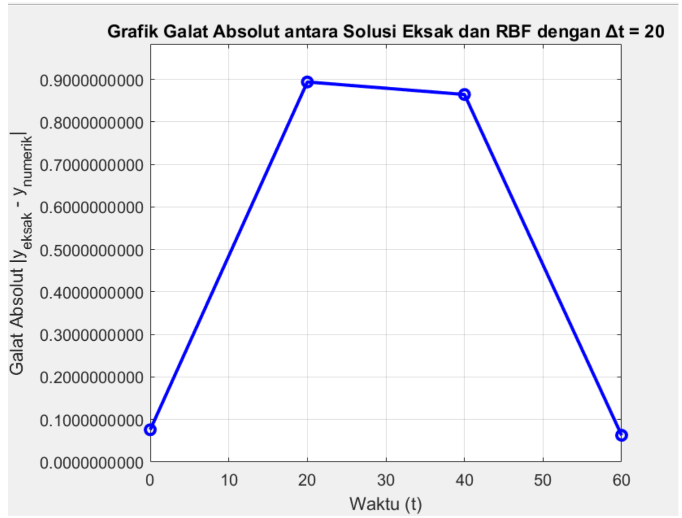

Fig. 7. Error Graph of String Vibration Solution $y(t)$ for $t \in [0,60]$ with initial conditions $y(0) = 12.25$ and $\dot{y}(0) = 1$ with parameters $\Delta t = 20$; $K = 1000$; $m = 2500$; $\delta = 0.01$; $g = 9.8$

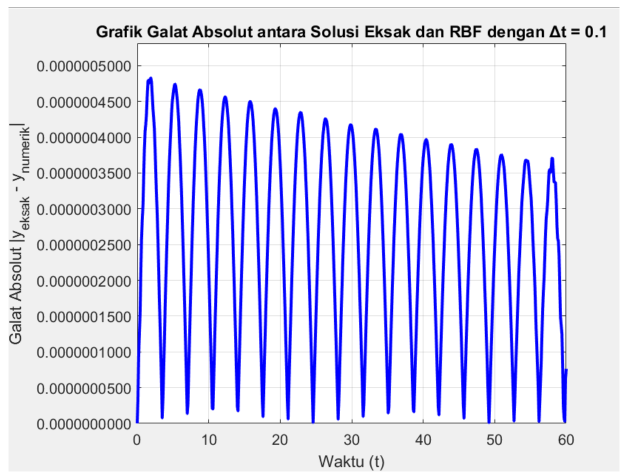

Fig. 8. Error Graph of String Vibration Solution $y(t)$ for $t \in [0,60]$ with initial conditions $y(0) = 12.25$ and $\dot{y}(0) = 1$ with parameters $\Delta t = 0.1$; $K = 1000$; $m = 2500$; $\delta = 0.01$; $g = 9.8$

Based on the graphs in Fig. 7 and Fig 8, the absolute error of the numerical solution shows an oscillation pattern that decreases over time. At $\Delta t = 20$ (Fig. 7), the initial error is around 0.1, then increases to a peak of around 0.9 at $t = 20$, and begins to decrease again to approach 0 at $t = 60$. Meanwhile, at $\Delta t = 0.1$ (Fig. 8), the initial error is very small, around 0.0000005, and oscillates with low amplitude all the time. This pattern shows that the RBF method is able to approach the exact solution accurately, especially with small time steps, and has numerical stability in modeling oscillatory systems.

IV. CONCLUSION

Numerical solutions of mathematical models of string vibrations solved using the Radial Basis Function (RBF) method show oscillation patterns that are close to the exact solution. At time discretization with $\Delta t = 0.1$, the RBF method is able to produce smooth, continuous solutions that consistently follow the dynamics of the system described by the exact solution. Meanwhile, at $\Delta t = 20$, although the number of evaluation points is smaller, the main oscillation pattern can still be captured well by the RBF method.

Analysis of the absolute error for both Δt values shows that the error oscillates but remains within a controlled range. At $\Delta t = 0.1$, the error is very small, indicating a consistent oscillation close to 0. While at $\Delta t = 20$, the error tends to be larger and fluctuates, but still shows a decreasing trend at the end of the interval. Overall, the RBF method is proven to be effective in solving second-order differential equations in the string vibration model, and can be used as an accurate alternative numerical approach to the exact solution.

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